ALGEBRAS OF HOLOMORPHIC FUNCTIONS ON ONE-DIMENSIONAL VARIETIES

BY

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1. Introduction. By C^n we mean n-dimensional complex vector space. For $z \in C^n$, we will represent the ring of germs of holomorphic functions at z by Θ_z^n, and the sheaf of germs of holomorphic functions in C^n by Θ^n (for sheaf-theoretic terminology, see [2; 5]). In general, a *ringed space* is a pair (X, Θ), where X is a locally compact Hausdorff space and Θ is a sheaf of rings of germs of continuous functions on X (see [6]). Thus (C^n, Θ^n) is a ringed space. By *holomorphic function* on the ringed space (X, Θ) we mean a continuous function on X whose germ at any $x \in X$ is in Θ_x. If U is an open subset of X, $(U, \Theta | U)$ is a ringed space; we will let $H(U, \Theta)$ denote the ring of holomorphic functions on U. Where it is clear what Θ is we will write $H(U)$ for $H(U, \Theta)$.

Let U be an open set in C^n, and $f_1, \cdots, f_t \in H(U, \Theta^n)$. Let $V = \{z \in U; f_i(z) = 0, 1 \leq i \leq t\}$. We will write $V = V(f_1, \cdots, f_t)$. We consider Θ^n as defining a sheaf of germs of continuous functions on V, and write $\Theta^V = \Theta^n | V$. Then (V, Θ^V) is a ringed space. A function f defined on V is thus holomorphic on V if, for every $x \in V$, there is a neighborhood W of x in C^n such that $f | W \cap V$ is the restriction to $W \cap V$ of a function holomorphic in W. Any closed subset of a domain U in C^n which is locally given as the zeros of a finite number of holomorphic functions is called a *variety*.

Let (X, Θ), (Y, Φ) be ringed spaces and $\phi \colon X \to Y$ a continuous map. By $\hat{\phi}(\Phi)$ we mean the sheaf on X whose stalk at $x \in X$ is the set $\{f \circ \phi; f \in \Phi_{\phi(x)}\}$. If $\hat{\phi}(\Phi) \subset \Theta$, we say ϕ is *holomorphic*. If ϕ is a homeomorphism and ϕ^{-1} is also holomorphic, we say ϕ is biholomorphic.

A *complex manifold* is a ringed space (X, Θ) with the following property: for every $x \in X$, there is a neighborhood U of x and a biholomorphic map of (U, Θ) onto (W, Θ^n), where W is an open set in C^n. An *analytic space* is a ringed space (X, Θ) with the following property: for every $x \in X$, there is a neighborhood U of x and a biholomorphic map of (U, Θ) onto (V, Θ^V), where (V, Θ^V) is a variety.

Let (S, Θ) be an analytic space. A point $x \in S$ is *regular* if x has a neighbor-

hood U such that (U, Θ) is a complex manifold. The set S_{reg} of regular points of S forms a complex manifold dense in S. By a one-dimensional analytic space we mean an analytic space S such that S_{reg} is a one-dimensional manifold. In this case $S_{sing} = S - S_{reg}$ is a discrete set of points on S (see [2]).

If K is a compact subset of an analytic space, let $H(K)$ be the ring of all functions holomorphic in a neighborhood of K. Let $A(K)$ be the uniform closure on K of $H(K)$. $A(K)$ is a Banach algebra with the uniform norm. Our main purpose is to study $A(K)$ with respect to the property of being a maximal subalgebra of $C(\Gamma(A(K)))$, the algebra of all continuous functions on the Silov boundary of $A(K)$. This will be done under the restriction that K consists of a domain bounded by finitely many piecewise analytic curves. It has been shown, in the case where S is a Riemann surface, that $A(K)$ is maximal in $C(\Gamma(A(K)))$ (Wermer, Royden [9; 10]). This result is our starting point. In particular, we will associate with any one-dimensional analytic space S a one-dimensional complex manifold R, and a projection $\pi: R \rightarrow S$, and with compact K on S, a compact $\tilde{K}$ on R. We shall show that, in a natural sense, $A(K)$ is a subalgebra of $A(\tilde{K})$ of finite codimension in $A(\tilde{K})$; this fact is the basis of the discussion.

We shall also attempt to describe certain subalgebras of $A(K)$, for K a compact subset of a Riemann surface. These subalgebras are of the form $A_0(K, U)$, the uniform closure on K of a subalgebra A_0 of $H(U)$, where U is a neighborhood of K. The first result in this direction is Wermer's theorem.

1.1. THEOREM (WERMER [11]). *Let R be a Riemann surface, Γ a simple closed analytic curve on R such that Γ is the boundary of a domain D with $D \cup \Gamma$ compact. Let f, g be holomorphic in a neighborhood of $D \cup \Gamma$, df does not vanish on Γ and f, g together separate points on Γ. Then there is a finite subset T of $D \cup \Gamma$ and an integer n such that if h is in $A(D \cup \Gamma)$, and h vanishes at each point of T with order no less than n, then h is approximable on $D \cup \Gamma$ by functions in the algebra $\{f, g\}$ of polynomials in f and g.*

In the appendix of [11] he extends this to finitely many functions. The crucial feature of this theorem is to prove that $D \cup \Gamma$ is $\{f, g\}$-convex (see the definition below); with this assumption the proof depends only on theorems of several complex variables of a general nature.

Later Bishop proved a closely related, but essentially different theorem.

DEFINITION. Let U be an open set on an analytic space S, and K a compact set in U. Let A_0 be a subalgebra of $H(U)$. We say K is A_0-*convex* if $K = \{x \in U; |f(x)| \leq \sup\{|f(y)|; y \in K\}$ for all $f \in A_0\}$. For $V \subset U$ open, we say V is A_0-*convex* if V can be written as the increasing union of a sequence of A_0-convex compact sets.

1.2. THEOREM (BISHOP [3]). *Let R be an open Riemann surface, K a compact subset of R, B a subalgebra of $H(R)$ such that (i) K is B-convex, (ii) $\{p \in R;$*

there is a $q \neq p$ in R such that $f(p) = f(q)$ for all $f \in B$ $\} \cup \{p \in R; df(p) = 0$ *for all* $f \in B\}$ *(the singular set relative to B, denoted by $S(B)$) is finite.*

Then there is a positive integer N such that $\overline{B}$, the closure of B on K, contains the ideal of $C(K) \cap H(\text{int } K)$ consisting of those functions which vanish on $S(B) \cap K$, and vanish of order at least N at those points of $S(B)$ interior to K.

The main point here is the approximation theorem, which we shall use in the maximality discussion. In §6 we shall prove the following theorem.

6.7. Theorem. *Let K be a compact subset of a Riemann surface R, and let U be a neighborhood of K. Suppose A^0 is a subalgebra of $H(U)$ such that* (i) *A^0 is generated by $f_1, \cdots, f_n \in H(U)$,* (ii) *$K$ is A^0-convex,* (iii) *$\{p \in U$; there is a $q \in U$ such that $p \neq q$ but $f(p) = f(q)$ for all $f \in A^0\}$ is finite. Let A be the uniform closure of A^0 on K.*

Then there is a variety V in the unit polycylinder of some C^t and a map ϕ of a neighborhood of K onto V such that

$$A = \{f \circ \phi; f \text{ in the closure of the polynomials on } \phi(K)\}.$$

Together these three theorems give a complete description of $A_0(K, U)$ when it is finitely generated.

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2. Ideals of $A(K)$ on a Riemann surface. Let R be a compact Riemann surface, and S any subset of R. We write $M(S)$ for the vector space of functions, meromorphic on R which are regular on S. From the Riemann-Roch theorem we obtain the following result.

2.1. Lemma. *Let U be any open set on R.*

(i) *Given $p \neq q$ in U, there is an $f \in M(U)$ such that $f(p) \neq f(q)$.*

(ii) *Given $p, p_1, \cdots, p_t$ in U, and positive integers $n, n_1, \cdots, n_t$, there is an $f \in M(U)$ with zeros of orders $\geq n_i$ at p_i, $1 \leq i \leq t$ and a zero of order n at p (i.e., if (U, z) is a local parameter at p, f has the Taylor series in $U: f = \sum a_n z^n$ with $a_i = 0$, $i < n$, and $a_n \neq 0$).*

Proof. (i) Let $p \neq q$ in U; by the Riemann-Roch theorem, there is a meromorphic function f regular on U with a zero of some order k at p. If $f(q) \neq 0$, then $f(p) \neq f(q)$. Suppose f has a zero of order m at q. Let g be meromorphic on R, with a pole only at q of order, say, n. Then $f^n g^k(q) \neq 0$, but $f^n g^k(p) = 0$, so $f^n g^k$ is the required function.

(ii) Let now $f \in M(U)$ such that f has a zero of order at least n_i at p_i, and is zero at p. Let k be the order of the zero at p. Again by the Riemann-Roch theorem there is a meromorphic function g whose only pole is at p and is of order m, where m is prime to p, so long as m is large enough. Then

there are positive s, t such that $sk - tm = 1$. Then $f^s g^t$ is in $M(U)$ and has a zero of order 1 at p. Thus $(f^s g^t)^n$ has a zero of order n at p, and obviously still has zeros of orders at least n_i at p_i.

Now let K be a compact proper subset of R. $M(K)$ is contained in $H(K)$, so the closure $A(K)$ of $H(K)$ in the uniform norm on K contains the restriction to K of functions in $M(K)$. We use this fact to show that the structure space $S(A(K))$ of $A(K)$, (the set of complex holomorphisms of $A(K)$) is just K.

2.2. LEMMA. $S(A(K)) = K$.

Proof. By Lemma 2.1, M separates the points of K; thus $A(K)$ separates points of K. Every point of K thus determines a continuous homomorphism of $A(K)$, so $S(A(K))$ contains K.

Let U be a neighborhood of K. Let h be in $S(A(K))$, different from evaluation at some point r in K. Let F be meromorphic on R with its only pole in U a simple pole at r. Now there is a g in $H(K)$ such that $h(g) = 1$, $g(r) = 0$. Then gF is in $H(K)$. Let $t = h(gF)$; then for $P = F - t$, gP is in $M(K)$ and $h(gP) = h(g(F - t)) = h(gF) - th(g) = 0$. Let $r_1, \cdots, r_n$ be the zeros of P on K of order $m_1, \cdots, m_n$, respectively. If h is not evaluation at any r_j, there are f_j in $H(K)$, $h(f_j) = 1$, $f_j(r_j) = 0$. Let $f = (\prod f_j^{m_j}) P^{-1}$. f is in $H(K)$ and, since $f(r) = 0$, fP is also in $H(K)$. Thus $h(\prod f_j^{m_j}) = \prod [h(f_j)]^{m_j} = 1$; but $\prod f_j^{m_j} = fP$, so $h(\prod f_j^{m_j}) = h(fP) = h(fP)h(g) = h(fPg) = h(f)h(gP) = 0$, a contradiction. Thus h is evaluation at r or some r_j, i.e., evaluation at some point of K.

This proof is essentially a part of Arens' proof that the algebra $C(K) \cap H(\text{int } K)$ has K as its structure space [1].

We now exploit the infinite dimensionality of $M(K)$ as a vector space to obtain a theorem which will be useful in the discussion of holomorphic functions on one-dimensional analytic spaces.

2.3. THEOREM. *Let* $x_1, \cdots, x_n \in K$. *Let* (U_i, z_i) *be a local coordinate at* x_i, *and let* $P_i(X)$ *be polynomials of degree* n_i *over the complex numbers. Then there is a* $g \in M(K)$ *such that the first* n_i *terms of the Taylor expansion of* g *about* x_i *in* z_i *agrees with* $P_i(z_i)$.

Proof. For $f \in M(K)$, we can expand f in a Taylor series in z_i about x_i: $f = \sum \lambda_{ij}(f) z^j$. $\lambda_{ij}(f)$ is a linear functional on $M(K)$. According to 2.1 (ii) we can find an $f_{ij} \in M(K)$ such that f_{ij} has a zero of order j at x_i and a zero of order at least n_r at x_r. That is, $\lambda_{rs}(f_{ij}) = 0$ if $r \neq i$ and $s \leq n_r$, or if $r = i$ and $s < j$. Thus the matrix $M = (\lambda_{rs}(f_{ij}))$, $1 \leq r$, $i \leq n$, $0 \leq s \leq n_r$, $0 \leq j \leq n_i$, is a $(\sum n_i) \times (\sum n_i)$ triangular matrix all of whose diagonal entries are nonzero. Thus M is nonsingular.

Write $P_i(X) = \sum a_{ij} X^j$. Let a be the vector (a_{ij}). Then there is a vector $b = (b_{ij})$ such that $Mb = a$. Let $g = \sum b_{ij} f_{ij}$; then $g \in M(K)$, and $\lambda_{rs}(g) = \sum b_{ij} \lambda_{rs}(f_{ij}) = a_{rs}$; thus g is the required function.

2.4. COROLLARY. *Let $x_1, \cdots, x_n$ be in the interior of K. Let $I = \{f \in A(K);$ f has a zero of order $\geq n_i$ at $x_i\}$. Then $A(K) = M(K) + I$.*

Proof. Let (U_i, z_i) be a local parameter at x_i, and let $f \in A(K)$. By 2.3, there is a $g \in M(K)$ such that the Taylor expansion of g in z_i at x_i coincides with that of f up to the n_ith term. Then $f - g \in A(K)$, and vanishes of order $\geq n_i$ at x_i, so $f - g$ is in I. Thus $f = g + (f - g)$ is in $M(K) + I$.

3. Normalization of an analytic space. Let S be a one-dimensional analytic space. We shall show that there is a complex manifold R and a projection $\pi: R \to S$ such that for U an open set on S, $\overline{U}$ compact, the set $\{f \circ \pi; f \in H(U)\}$ is of finite codimension in $H(\pi^{-1}(U))$. R will be the *normal model* for S (see [6]).

DEFINITION. Let S be an analytic space. The *normalization* of S is a pair (R, π) such that: (i) R is a locally compact, locally connected Hausdorff space; (ii) $\pi: R \to S$ is continuous, proper and onto; (iii) $\pi^{-1}(s)$ is finite for all s in S (in fact, $\pi^{-1}(s) = \{p_1, \cdots, p_{k(s)}\}$, where $k(s)$ is the number of irreducible components of S at x); (iv) $\pi^{-1}(S_{\text{sing}})$ separates no connected set in R and is nowhere dense; (v) $\pi: R - \pi^{-1}(S_{\text{sing}}) \leftrightarrow S - S_{\text{sing}}$ is a homeomorphism.

The points of R are pairs (s, V), $s \in S$ and V an irreducible branch of S at s, and $\pi(s, V) = s$. The basis neighborhoods are of the form $O(U, V) = \pi^{-1}(U \cap V)$ where V is an irreducible branch of S in the open set U. The space R is called the *normal model* of S.

Let $U \subset R$ be open. If $f: U \to C$ is continuous and $f \circ \pi^{-1} | U - \pi^{-1}(S_{\text{sing}})$ is holomorphic, we say that f is holomorphic on U. Let Θ^R be the sheaf of germs of holomorphic functions on R. Then π is a homomorphic map.

The following facts were proven by Oka [8; 6].

3.1. THEOREM. *The normal model exists, is an analytic space and is unique up to a biholomorphic map with respect to the properties (i)–(v) and that π is holomorphic.*

3.2. THEOREM. *If $s \in S$ is an irreducible point, $\pi^{-1}(s) = p$. Θ_p^R is the integral closure of $\pi(\Theta_s^S) = \{f \circ \pi; f \in \Theta_s^S\}$. There is a nonzero ideal C of Θ_p^R such that $C\Theta_p^R \subset \pi(\Theta_s^S)$.*

We now prove

3.3. THEOREM. *The normal model of a one-dimensional analytic space is a one-dimensional complex manifold.*

Proof. We know $R - \pi^{-1}(S_{\text{sing}})$ is a complex manifold, since it is biholomorphic to $S - S_{\text{sing}}$. Let $s \in S_{\text{sing}}$. Since S_{sing} is discrete, s has a neighborhood U such that $U \cap S_{\text{sing}} = \{s\}$. Let $V_1, \cdots, V_t$ be the branches of S in U; then $\pi^{-1}(U) = \cup O(U, V_i)$ and each $O(U, V_i)$ is connected. It is easy to see that $\pi: O(U, V_i) \leftrightarrow U \cap V_i$ is a homeomorphism and $\pi: O(U, V_i) - \{p_i\} \leftrightarrow U \cap V_i - \{s\}$ is biholomorphic, where $p_i = (s, V_i)$. We need only prove that we can

define a local coordinate at p_i which, with the natural structure of $O(U, V_i)$ $-\{p_i\}$, makes of $O(U, V_i)$ a Riemann surface such that π is still holomorphic. Then, by the uniqueness, this structure is the normalization structure. Thus, since we are interested in only $O(U, V_i)$ near p_i, we can assume S has only one branch at s, further, that S is embedded as a variety V in a neighborhood of the origin ($s=0$) in C^n.

3.5. LEMMA. *Let V be an irreducible one-dimensional variety in C^n at 0, and suppose that 0 is the only singular point of V. Then there are a Riemann surface R^1 and a holomorphic homeomorphism $\rho: R^1 \leftrightarrow V$ such that $\rho \mid R^1 - \rho^{-1}(0)$ is biholomorphic.*

Proof. In a neighborhood of 0, by the Nullstellensatz [5, Chapter XIV] for a suitable choice of coordinates we can write $V - \{0\} = V(f_1, \cdots, f_{n-1})$ where

$$f_1(z_1, \cdots, z_n) = g_1(z_1, z_2) = z_2^r + \sum_{i=0}^{r-1} a_i(z_1)z_2^i,$$

$$f_j(z_1, \cdots, z_n) = z_j - \frac{g_j(z_1, z_2)}{D(z_1)},$$

where g_j are distinguished polynomials, and $D(0)=0$.

Let R^1 be the Riemann surface covering $\{|z_1| < \epsilon\}$ of the algebraic function defined by $g_1(z_1, z_2) = 0$, and $\pi^0: R^1 \to \{|z_1| < \epsilon\}$ the projection. Let x in R^1, $\pi^0 x \neq 0$; then x corresponds to a certain branch ϕ_x of this algebraic function. We define $\rho: R^1 \to V \cap \{|z_1| < \epsilon\}$:

$$\rho(x) = \left(\pi^0 x, \phi_x(\pi^0 x), \frac{g_2(\pi^0 x, \phi_x(\pi^0 x))}{D(\pi^0 x)}, \cdots, \frac{g_n(\pi^0 x, \phi_x(\pi^0 x))}{D(\pi^0 x)}\right),$$

$$\rho(x) = 0 \quad \text{if} \quad \pi^0 x = 0.$$

If π^1 is the projection of V onto $\{|z_1| < \epsilon\}$ it is obvious that $\pi^0 = \pi^1 \circ \rho$.

ρ is one-one. For if $\rho x = \rho y$, then $\pi^0 x = \pi^0 y$, $\phi_x(\pi^0 x) = \phi_y(\pi^0 y) = \phi_y(\pi^0 x)$ which implies that $\phi_x = \phi_y$, or $x = y$.

ρ is biholomorphic on $R^1 - \rho^{-1}(0)$. Let $\pi^0 x \neq 0$; then for U_x a small enough neighborhood of x, (U_x, π^0) is a local parameter at x. Similarly, since πx is in V_{reg}, $(U_{\rho x}, \pi^1)$ is a local parameter at x for $U_{\rho x}$ small enough. But since $\pi^0 = \pi^1 \circ \rho$, ρ also is biholomorphic in a sufficiently small neighborhood of x.

Easily ρ is holomorphic. For if f is holomorphic on V, $f \circ \rho$ is holomorphic on $R - \rho^{-1}(0)$, but also continuous. Thus since R^1 is a complex manifold, $f \circ \rho$ is holomorphic on R^1.

Now, we can transfer the structure of R^1 to R, the normalization of V by $\pi^{-1} \circ \rho$, making of R a complex manifold. This structure coincides with the given one on $R - \pi^{-1}(0)$, and with it π is still holomorphic. Thus the two structures do coincide and R is a complex manifold.

If x in S is a singular point, $\pi^{-1}(x) = \{r_1, \cdots, r_n\}$ and for U a neighborhood of x, there are neighborhoods U_i of r_i such that $\pi: \bigcup U_i$ onto U and $\pi: \bigcup U_i - \{r_1, \cdots, r_n\} \to U - \{x\}$ one-one and onto. We call the $\pi(U_i)$ the *sheets* of S at x. Note that $\pi(U_i)$ is an analytic space and U_i is its normal model. If x is in $\pi(U_i)_{\text{reg}}$, then $\pi: U_i \leftrightarrow \pi(U_i)$ is a biholomorphic map. If x is in $\pi(U_i)_{\text{sing}}$, then π is just holomorphic. This type of point we call a *branch point* of $\pi(U_i)$ or a branch point of S if $\pi^{-1}(x)$ is just one point (x is an irreducible point). If $\pi^{-1}(x)$ is more than one point we call x an *identification point*.

Let x be a branch point of S; then $\pi: R \to S$ is one-one holomorphic from a neighborhood of $\pi^{-1}(x) = r$ to a neighborhood of x in S. For the moment we replace these neighborhoods by R and S respectively and we assume S is a variety V at 0 in C^n. Then $\hat{\pi}(\Theta_x^V)$ (the transfer by π of all functions holomorphic in a neighborhood of 0) is a subring of Θ_r^R. In fact we have:

3.6. **Lemma.** $\hat{\pi}(\Theta_x^V)$ *is precisely the set of germs of functions which can be uniformly approximated in a neighborhood of r by polynomials in the $\hat{\pi}(z_1), \cdots, \hat{\pi}(z_n)$. Further, $\hat{\pi}(\Theta_x^V)$ contains a power of the maximal ideal of Θ_r^R.*

Proof. Let f be in $\hat{\pi}(\Theta_x^V)$; then $f = g \circ \pi$, g in Θ_x^V. Then, for a sufficiently small polycylinder P^n about x, there is an h in $H(P^n)$ such that $g = h \mid V$. But then h is approximable by polynomials in compact neighborhoods of x, thus also is g; thus $g \circ \pi = f$ is approximable by polynomials in the $z_j \circ \pi = \hat{\pi}(z_j)$ in a neighborhood of r. Conversely, if f is in Θ_r^R, in a neighborhood of r, $f \circ \pi^{-1}$ is defined, since π is one-one. If f is approximable by polynomials in the $\hat{\pi}(z_j)$, $f \circ \pi^{-1}$ is approximable in a neighborhood of x on V by polynomials in the z_j. But Θ_x^V is closed in the topology of local uniform convergence [5], so $f \circ \pi^{-1}$ is in Θ_x^V, implying $f = f \circ \pi^{-1} \circ \pi$ is in $\hat{\pi}(\Theta_x^V)$.

Now, by 3.2, there is an ideal $\mathcal{C}$ of Θ_x^V such that $\mathcal{C}\Theta_r^R \subset \Theta_x^V$. But $\mathcal{C}\Theta_r^R$ is an ideal of Θ_r^R; thus Θ_x^V contains an ideal I of Θ_r^R, and $I \neq 0$, since $\mathcal{C} \neq 0$. But the only ideals of Θ_r^R are the powers of the maximal ideal. Let g be in I; suppose g has a zero of order n at r, g' has a zero of order $\geq n$. Then g'/g is in Θ_r^R, so $g' = g(g'/g)$ is also in I. Thus I contains M^n ($M = $ maximal ideal). Let $n^0 = \min\{n; I \supset M^n\}$. Then $I = M^{n^0}$; for if not then there is a g in I with a zero of order $m < n^0$, but this implies, as we have seen, $I \supset M^m$, contradicting the definition of n^0.

3.7. **Corollary.** *If x is a branch point of S, $\pi: R \to S$, $\pi(r) = x$, then there exists an N such that, if f is analytic in a neighborhood of r and vanishes at r with order $\geq N$, then f is in $\hat{\pi}(\Theta_x^S)$.*

Now we look at the local behavior at an identification point; so we may assume we have maps $\pi_i: \{|z| < 1\} \leftrightarrow V_i \subset P^n$, $\pi_i(0) = 0$, and $V_1, \cdots, V_k$ are the irreducible components of a variety V containing 0. Let $T = \{(f_1, \cdots, f_k); f_j \text{ in } \Theta_0^1 \text{ such that there exists } g \text{ in } \Theta_0^n, g \mid V_j \circ \pi_j = f_j\}$.

3.8. LEMMA. *There are integers $N_j \geq 0$ such that if f_j is in Θ_0^1 and vanishes at 0 with order $\geq N_j$, then $(f_1, \cdots, f_k)$ is in T.*

Proof. Write $V_i = V(p_1^i, \cdots, p_{n-1}^i)$. Since $V_j \cap V_i = \{0\}$, there is p_i^j in $I(V_j)$ such that $p_i^j|_{V_i}$ has its only zero at 0. Let n_{ij} be the order of vanishing of $p_i^j|_{V_i} \circ \pi_i$ at 0. According to 3.7 there is an integer n_{ii} such that if f is in Θ_0^1 and vanishes at 0 with order $\geq n_{ii}$, then f is in $\hat{\pi}_i(\Theta_0^n)$. Let $N_i = \sum_{j=1}^n n_{ij}$. These are the required integers.

Let $(f_1, \cdots, f_k)$ be as in the statement of the lemma. Then $f_i^1 = [f_i[\prod_{j \neq i} p_i^j|_{V_i} \circ \pi_i]^{-1}]$ is in Θ_0^j and vanishes with order $\geq n_{ii}$, so there is a g_i in Θ_0^n such that $g_i|_{V_i} = f_i^1$, or, what is the same, $f_i = (g_i \prod_{j \neq 1} p_i^j|_{V_i}) \circ \pi_i$. But now, for $m \neq i$, $g_m \prod_{j \neq m} p_m^j|_{V_i}$ vanishes on V_i, so we have

$$f_i = \left(\sum_{m=1}^k g_m \prod_{j \neq m} p_m^j \right) \circ \pi_i.$$

Then $g = \sum_{m=1}^k g_m \prod_{j \neq m} p_m^j$ is in Θ_0^n, $g \circ \pi_i = f_i$, proving $(f_1, \cdots, f_n)$ is in T.

Now consider K a proper compact set on a connected one-dimensional analytic space S. Since S_{sing} is discrete $K \cap S_{\text{sing}}$ is finite. Furthermore we can find a neighborhood U of K such that $\overline{U}$ is compact and $U \cap S_{\text{sing}} = K \cap S_{\text{sing}}$.

Let (R, π) be the normalization of S. Then $\pi^{-1}(U)$ has finitely many connected components. For any regular point on S can be connected to a singular point by an arc contained in S_{reg}, the set of regular points. Since π^{-1} is homeomorphic on S_{reg}, any $r \in R - \pi^{-1}(S_{\text{sing}})$ can be connected by an arc in $\pi^{-1}(U)$ to a point in $\pi^{-1}(S_{\text{sing}})$, but $\pi^{-1}(S_{\text{sing}} \cap U)$ is finite.

Thus we can write $\pi^{-1}(U) = \cup U_i$, where U_i is an open connected set on a Riemann surface R_i (a component of R). Since π is proper and $\overline{U}$ is compact, each $\overline{U}_i$ is compact, so we may consider U_i to be an open subset of a compact Riemann surface R_i'.

Now if $x \in K \cap S_{\text{sing}}$, $\pi^{-1}(x) = \{r_1, \cdots, r_t\}$. Now any r_i may be the only point of $\pi^{-1}(K)$ on that component of R, for the branch corresponding to r_i may intersect K only at x. We want to exclude such points from $\pi^{-1}(K)$, so instead we take $\tilde{K}$, the closure of $\pi^{-1}(K \cap S_{\text{reg}})$, and thus if K has no isolated points (which case we exclude) π maps $\tilde{K}$ onto K. Since $\tilde{K}$ differs from $\pi^{-1}(K \cap S_{\text{reg}})$ by a finite point set, we have the following: (a) if K is the closure of its interior, so also is $\tilde{K}$ the closure of its interior; (b) if int K is connected, int $\tilde{K}$ is connected; (c) if K is bounded by finitely many piecewise analytic curves, so is $\tilde{K}$.

Now $\tilde{K}$ is a disjoint union of finitely many proper compact sets $\tilde{K}_i$ on compact Riemann surfaces. Thus $A(\tilde{K})$ obviously is the direct sum of the $A(\tilde{K}_i)$, and so by 2.2, $S(A(\tilde{K})) = S(A(\tilde{K}_i)) = \cup \tilde{K}_i = \tilde{K}$.

3.9. THEOREM. *Let K be a compact subset of a connected one-dimensional analytic space S. Then there are finitely many compact Riemann surfaces*

$R_1, \cdots, R_t$ and $\tilde{K}_i \subset U_i \subset R_i$, $\tilde{K}_i$ compact, U_i open, and a map $\pi: \cup U_i$ onto a neighborhood U of K such that (i) for $R = \cup U_i$, (R, π) is the normalization of U, (ii) $\tilde{K} = \cup \tilde{K}_i$ is the closure of $\pi^{-1}(K \cap S_{\mathrm{reg}})$. (iii) Let $K \cap S_{\mathrm{sing}} = U \cap S_{\mathrm{sing}} = \{s_1, \cdots, s_n\}$. Let $\pi^{-1}(s_i) \cap \tilde{K} = \{x_{i1}, \cdots, x_{it_i}\}$. There exist integers n_{ij} such that $A(K) \supset \{f \circ \pi^{-1}; f \text{ holomorphic in a neighborhood of } \tilde{K}, f(x_{ip}) = f(x_{iq}) \text{ for all } p, q, 1 \leq i \leq n, f(x) - f(x_{ij}) \text{ has a zero of at least order } n_{ij} \text{ at } x_{ij}\}$. (iv) In particular, $A(K) \supset \{f \circ \pi^{-1}; f \text{ in } I\}$ where I is an ideal of $A(K)$ with finite hull; in fact hull $I = \pi^{-1}(K \cap S_{\mathrm{sing}})$.

Proof. If in the preceding discussion we take $\tilde{K}_i = \tilde{K} \cap U_i$, then (i), (ii) are verified. As for (iii), the integers n_{ij} are just those given by Lemma 3.8 applied at the point s_i. That is, if $\pi^{-1}(s_i) = \{x_{i1}, \cdots, x_{it_i}\}$, then for S_j a branch of S at s_i, let n_{ij} be the integer given by Lemma 3.8, so that if f_j is in $\Theta^R_{x_{ij}}$ and vanishes at x_{ij} with order $\geq n_{ij}$, then there is a $g \in \Theta^n_{s_i}$ (where we now consider a neighborhood of s_i as embedded in $\mathbb{C}^n$) such that $(g | S_j) \circ \pi = f_j$. Now let f be in $H(V, \Theta^R)$, V a neighborhood of $\tilde{K}$; suppose f is as described in (iii). Since f identifies the points which π identifies, $f \circ \pi^{-1}$ is a continuous function on K. Since $S_{\mathrm{sing}} \cap K = S_{\mathrm{sing}} \cap U$, $g = f \circ \pi^{-1}$ is a continuous function on $\pi(V)$ and since $\pi | R - \pi^{-1}(S_{\mathrm{sing}})$ is biholomorphic, for all regular points s we have $g_s \in \Theta^S_s$. If $s = s_i$, then by Lemma 3.8, since f is as described in (iii), there is a $G \in \Theta^n_{s_i}$ such that $G | S_j = g - g(s_i) | S_j$; so $g = G + g(s_i)$. But $G + g(s_i)$ is in $\Theta^n_{s_i}$, so g is in $\Theta^S_{s_i}$. Thus g is in $H(\pi(V), \Theta^S)$, so that $g | K = f \circ \pi^{-1} | K$ is in $A(K)$. Thus (iii) is proven.

Letting $H(\tilde{K}) = [f \text{ holomorphic in a neighborhood of } \tilde{K}]$, then by (iii), $A(K) \supset [f \circ \pi^{-1}; f \text{ in } I]$ where $I = [f \text{ in } H(\tilde{K}); f(x_{ij}) = 0, f \text{ has a zero of order } \geq n_{ij} \text{ at } x_{ij}]$ is an ideal of $H(\tilde{K})$. Now $H(\tilde{K})$ is dense in $A(\tilde{K})$. Thus $\bar{I}$ (closure of I in $A(\tilde{K})$) is also an ideal of $A(\tilde{K})$, and since $A(K)$ is closed, $A(K) \supset \{f \circ \pi^{-1}; f \text{ in } \bar{I}\}$. Now hull $\bar{I} = \{x_{ij}\}$. Surely each x_{ij} is in hull $\bar{I}$. On the other hand, if x is in $\tilde{K}$, $x \neq x_{ij}$, by 2.3, there is an f_{ij} in $H(\tilde{K})$ such that $f_{ij}(x) \neq 0$, $f_{ij}(x_{ij}) = 0$. Then $f = [\prod f_{ij}]^{\mathrm{sup}\, n_{ij}}$ is in I and $f(x) = 0$, so x is not in hull $\bar{I}$.

3.10. Theorem. $S(A(K)) = K$.

Proof. Since $A(K)$ is an algebra of continuous functions on K, evaluation at any point of K is a continuous homomorphism of $A(K)$. We prove $A(K)$ separates point of K. Let s, t be in K, s in S_{reg}. Let $x = \pi^{-1}(s)$. There is a function f_{tk} in $H(\tilde{K})$ such that $f_{tk}(x) \neq 0$, $f_{tk}(y_k) = 0$, for all y_k in $\pi^{-1}(t)$. There is an f_{ij} in $H(\tilde{K})$ such that $f_{ij}(x) \neq 0$, $f_{ij}(x_{ij}) = 0$. Then $f = [\prod f_{tk} f_{ij}]^{\mathrm{sup}\, n_{ij}}$ is in I and $f(x) \neq 0$, $f(\pi^{-1}t) = 0$. Thus $g = f \circ \pi^{-1}$ is in $A(K)$ and $g(s) \neq g(t)$.

Suppose now $s = s_1$, $t = s_2$. By 2.3 there is an f in $M(\tilde{K})$, $f(x_{1j}) = 1$, $f(x_{ij}) = 0$ if $i \neq 1$, and $f(x) - f(x_{ij})$ has a zero of order at least n_{ij} at x_{ij}. Then f is in $H(K)$ and by (iii) of 3.9, $g = f \circ \pi^{-1}$ is in $A(K)$. But $g(s) = 1$, $g(t) = 0$. Thus $A(K)$ separates points.

Let h be in $S(A(K))$, suppose h is not evaluation at any $s_1, \cdots, s_n$. Then there is an f_1 in $A(K)$ such that $f_1(s_j) = 0$, $1 \leq j \leq n$, and $h(f_1) = 1$. For f in $\bar{I}$, let $\bar{h}(f) = h(f \circ \pi^{-1})$. $\bar{h}$ is a nonzero continuous homomorphism on $\bar{I}$, since $[f_1 \circ \pi]^N$, where $N = \sup\{n_{ij}\}$ is in $\bar{I}$, and $\bar{h}((f_1 \circ \pi)^N) = h((f_1)^N) = 1$. But now, by 2.2 $S(A(\tilde{K})) = \tilde{K}$, and hull $\bar{I} = \{x_{ij}\}$, so $S(\bar{I}) \equiv K - \{x_{ij}\}$. Then $\bar{h}(f) = f(x)$ for some x in K, $x \neq x_{ij}$. Thus πx is a regular point of S. Let g be in $A(K)$; we prove $h(g) = g(\pi x)$. Then, since $(f_1)^N \circ \pi$ is in I, $g(f_1)^N \circ \pi = (g \circ \pi)((f_1)^N \circ \pi)$ is in I, so that

$$h(g) = h(g)h((f_1)^N) = h(g(f_1)^N) = h(g(f_1)^N \circ \pi)$$
$$= g(\pi x)((f_1)^N(\pi x)) = g(\pi x)h((f_1)^N \circ \pi) = g(\pi x).$$

Then $S(A(K)) = K$.

4. Runge sets. If K is a compact set on an analytic space S, it is not necessarily true that $A(K) = C(K) \cap H(\text{int } K)$, i.e., it is not always true that every function continuous on K and analytic in the interior of K is approximable by functions holomorphic in a neighborhood of K. If S is a complex manifold, it is evident that $A(K) \subset C(K) \cap H(\text{int } K)$ since $H(\text{int } K)$ is closed in the topology of uniform convergence. But for S a general analytic space this has only recently been proven by Remmert and Grauert [5]. We have already used this result in 3.6. The possibility of $A(K) = C(K) \cap H(\text{int } K)$ depends greatly on the geometry of K.

DEFINITION. If $K \subset S$ is a proper compact subset of S and there is a domain $U \supset K$ such that K is $H(U)$-convex, we call K a *Runge set*.

4.1. THEOREM (RUNGE'S THEOREM) (BEHNKE). *Let R be a Riemann surface, $U \subset U'$ domains on R. Suppose U is simply connected with respect to U', i.e., if $\gamma \subset U$ bounds a domain in U' it bounds a domain in U. Then every function holomorphic on U can be approximated uniformly on compact sets by functions holomorphic on U' [3].*

4.2. COROLLARY. *Let K be a compact, proper subset of R. If there is a family of domains $U(n) \subset U$, n an integer, satisfying* (i) *$U(n)$ is simply connected with respect to $U(1)$,* (ii) *$U(n) \supset U(n+1)$,* (iii) *$\bigcap_n U(n) = K$, then $H(U(1))$ is dense in $A(K)$.*

Proof. Let f be in $A(K)$; f can be uniformly approximated by functions holomorphic in a neighborhood of K. If U is a neighborhood of K there is an n such that $U(n) \subset U$. If f is in $H^0(U, \Theta^R)$, then $f|_{U(n)}$ is uniformly approximable on K by functions in $H(U(1))$. Thus the corollary is proven.

If K has a neighborhood U such that $H(U)$ is dense in $A(K)$, then K is $H(U)$-convex. For we know that $S(A(K)) = K$, so it follows that $S(H(U)) = K$, with the sup norm on K on $H(U)$. Thus if $x \in U - K$, the map $f \rightarrow f(x)$ is not a continuous multiplicative functional on $H(U)$ in this norm, so there is an $f \in H(U)$ such that $|f(x)| > \|f\|_K$. But this is the condition of $H(U)$-convexity.

It is easily seen that if $K = \Omega \cup \gamma_1 \cup \cdots \cup \gamma_n$, Ω a domain, $\gamma_1, \cdots, \gamma_n$ piecewise analytic curves, then the hypotheses of 4.2 are satisfied with $U(1)$ a sufficiently small neighborhood of K, so K is a Runge set.

4.3. COROLLARY. *Let K be a Runge set in a compact Riemann surface R. There is a neighborhood U of K such that $H(U)$ is dense in $C(K) \cap H(\operatorname{int} K)$, and in particular $A(K) = C(K) \cap H(\operatorname{int} K)$.*

This follows from an immediate application of Bishop's theorem, Theorem 1.2. We now obtain the same result for Runge sets on one-dimensional analytic spaces.

4.4. THEOREM. *Let K be a compact subset of S, and $\tilde{K}$ as defined in §3. K is a Runge set if and only if $\tilde{K}$ is a Runge set. If K is a Runge set, $A(K) = C(K) \cap H(\operatorname{int} K)$.*

Proof. If K is a Runge set, there is a $U \subset S$ such that $U \supset K$ and K is $H(U)$-convex. If $K \subset V \subset U$, then obviously K is also $H(V)$-convex, so we may assume U chosen so small that $\pi^{-1}(U) \cap \pi^{-1}(S_{\text{sing}}) = \tilde{K} \cap \pi^{-1}(S_{\text{sing}})$. Then if x is in $\pi^{-1}(U)$, but not in $\tilde{K}$, then πx is not in K, so there is an $f \in H(U)$ such that $|f(\pi x)| > \|f\|_K$. Then $f \circ \pi$ is holomorphic on $\pi(U)$ and $|f \circ \pi(x)| > \|f \circ \pi\|_{\tilde{K}}$. Thus $\tilde{K}$ is $H(\pi^{-1}(U))$-convex. Now we can write $\tilde{K} = \cup \tilde{K}_i$, a disjoint union and $\pi^{-1}(U) = \cup \tilde{U}_i$ where $\tilde{U}_i \supset \tilde{K}_i$ are both subsets of a compact Riemann surface. Obviously $\tilde{K}_i$ is $H(\tilde{U}_i)$-convex, so it follows from 4.3 that $A(\tilde{K}_i) = C(\tilde{K}_i) \cap H(\operatorname{int} \tilde{K}_i)$, and then $A(\tilde{K}) = C(\tilde{K}) \cap H(\operatorname{int} \tilde{K})$.

Now we assume only $\tilde{K}$ is a Runge-set, and is $H(\tilde{U})$-convex. Let $K \cap S_{\text{sing}} = \{s_1, \cdots, s_n\}$, let $s_1, \cdots, s_e$ be in ∂K, $s_{e+1}, \cdots, s_n$ in int K. Let $\pi^{-1}(s_i) = \{x_{ij}, 1 \leq j \leq k_i\}$, and n_{ij} the integer corresponding to s_i as 3.9. Let $I = \{f \text{ in } H(\tilde{U}); f \text{ has a zero of at least order } n_{ij} \text{ at } x_{ij}\}$. Now $\tilde{K}$ is I-convex. For I is an ideal in $H(\tilde{U})$ and since $\tilde{K}$ is $H(\tilde{U})$-convex, the closure of $H(\tilde{U})$ on $\tilde{K}$ is $A(\tilde{K})$, and $S(A(\tilde{K})) = \tilde{K}$. Thus the closure $\bar{I}$ of I is an ideal of $A(\tilde{K})$, so $S(\bar{I})$ is a subspace of $\tilde{K}$. But if $\tilde{K}$ is not I-convex, there is an x in $\tilde{U} - \tilde{K}$ such that $|f(x)| \leq \|f\|_{\tilde{K}}$ for all f in I, so evaluation at x extends to a homomorphism of $\bar{I}$, not evaluation at any point of $\tilde{K}$; since I separates points of $\tilde{U}$ (but for hull $I \subset \tilde{K}$), contradicting $S(\bar{I}) \subset \tilde{K}$. Since $I \subset [f \circ \pi; \text{ in } H(\pi(\tilde{U}))]$, then K is $H(\pi(\tilde{U}))$-convex.

Now, by Bishop's theorem, 1.2, $\bar{I}$ contains $\{f \text{ in } C(K) \cap H(\operatorname{int} K) \text{ such that } f \text{ vanishes on } \pi^{-1}(S_{\text{sing}}) \text{ and has a zero of order } \geq N \text{ at } x_{ij}, i > e\}$, for some integer N. Let $\mathcal{g}$ be the ideal of $A(\tilde{K})$, $\mathcal{g} = \{f \text{ in } A(\tilde{K}); f(x) = f(x_{ij}) \text{ has a zero of order } \geq N \text{ for } i > e\}$.

Now let f be in $C(K) \cap H(\operatorname{int} K)$ such that f vanishes on $S_{\text{sing}} \cap \partial K$; then $f \circ \pi$ is in $C(\tilde{K}) \cap H(\operatorname{int} \tilde{K}) = A(\tilde{K})$, since $\tilde{K}$ is a Runge set. By 2.4 considering only the points $x_{ij}, i > e$, we can write $f \circ \pi = g + h$, g is meromorphic on R, holomorphic on K, and h is in $\mathcal{g}$. We can arrange, in fact, that h is in $\bar{I}$ by adding and subtracting a meromorphic function in $\mathcal{g}$ which takes the values $g(x_{ij})$ at $x_{ij}, i < e$; then h vanishes on $\pi^{-1}(S_{\text{sing}})$. Since g is holomorphic in a

neighborhood of $\tilde{K}$, we can write $g = g_1 + g_2$, where g_1, g_2 are also meromorphic functions, holomorphic on $\tilde{K}$, such that g_1 vanishes with order n_{ij} at x_{ij}, $i \leq e$, and g_2 is in $\mathcal{I}$, i.e., let g_2 be in $\mathcal{I}$ and g_2 have the same power series up to the n_{ij} term as $-g$ at x_{ij} (in some local coordinate system) for all $i \leq e$. This can be done by 2.3. Also g_1, g_2 vanish on $\pi^{-1}(S_{\text{sing}})$. Thus we have $f \circ \pi = g_1 + (g_2 + h)$; g_1 vanishes of order n_{ij} at x_{ij}, $i \leq e$, and $g_2 + h$ is in $\mathcal{I}$ and vanishes on $\pi^{-1}(S_{\text{sing}})$. Thus $g_2 + h$ is in $\bar{I}$ and $\{F \circ \pi^{-1}; F \text{ in } \bar{I}\} \subset A(K)$, so $(g_2 + h) \circ \pi^{-1}$ is in $A(K)$, so also is in $C(K) \cap H(\text{int } K)$. Thus $g_1 \circ \pi^{-1} = f - (g_2 + h) \circ \pi^{-1}$ is also in $C(K) \cap H(\text{int } K)$. But g_1 is meromorphic on R, and holomorphic on $\tilde{K}$, so is holomorphic in a neighborhood W of $\tilde{K}$. Thus $g_1 \circ \pi^{-1}|S_{\text{reg}}$ is in $H(\pi(W) \cap S_{\text{reg}})$. But for x_{ij}, $i \leq e$, g_1 vanishes of order n_{ij}, so by 3.8 $[g_1 \circ \pi^{-1}]_{s_i}$ is in $\Theta_{s_i}^{\mathcal{S}}$. This, for all $i \leq e$, but for $i > e$, $g_1 \circ \pi^{-1}$ is in $\Theta_{s_i}^{\mathcal{S}}$ since s_i is in int K. Thus $g_1 \circ \pi^{-1}$ is in $H(\pi(W), \Theta^S)$, $\pi(W)$ is a neighborhood of K, so $g_1 \circ \pi^{-1}$ is in $A(K)$. $(g_2 + h) \circ \pi^{-1}$ is in $A(K)$ also, thus $f = g_1 \circ \pi^{-1} + (g_2 + h) \circ \pi^{-1}$ is in $A(K)$.

Since $\partial K \cap S_{\text{sing}} = \{s_1, \cdots, s_e\}$ is finite, given any complex numbers $t_1, \cdots, t_e$, there is a g in $A(K)$ such that $g(s_i) = t_i$. Thus if f is in $C(K) \cap H(\text{int } K)$, there is a g in $A(K)$ such that $g(s_i) = f(s_i)$. Thus $f - g$ is in $C(K) \cap H(\text{int } K)$ and vanishes on $\partial K \cap S_{\text{sing}}$, so by the above is in $A(K)$. Thus $f = (f - g) + g$ is also in $A(K)$. Theorem 4.4 is proven.

We obtain as a corollary of the above (more explicitly, corollaries of Bishop's Theorem (1.2)) certain approximation theorems; more general results are found in [13].

4.5. Theorem. *Let γ be a piecewise analytic arc on a one-dimensional analytic space S. Then $A(\gamma) = C(\gamma)$, i.e., the algebra of functions holomorphic in a neighborhood of γ is dense in $C(\gamma)$.*

Proof. Let R be the normalization of S, $\pi: R \to S$, and let $\tilde{\gamma} = \pi^{-1}(\gamma \cap S_{\text{reg}})$. Then $\tilde{\gamma}$ is a piecewise analytic arc on R, and is obviously a Runge set. Then γ is a Runge set and 4.4 applies, so $A(\gamma) = C(\gamma)$, since int γ is empty.

Now since the image under a holomorphic map of a domain in C^1 into C^n is always an analytic space [9], we have

4.6. Theorem. *Let γ be any analytic arc in C^n (i.e., the image of $[0, 1]$ under a holomorphic homeomorphism). Then $A(\gamma) = C(\gamma)$.*

4.7. Theorem. *Let γ be a piecewise analytic arc on a variety V in a polynomial convex subset U of C^n. Then the polynomials are dense in $C(\gamma)$.*

Proof. Let R be the normalization of V, $\pi: R \to S$, and let $\tilde{\gamma} = \pi^{-1}(\gamma \cap S_{\text{reg}})$. We can easily write $\tilde{\gamma}$ as the decreasing intersection of domains which are simply connected in R. Thus, as in 4.2, $\tilde{\gamma}$ is $H(R)$-convex, and similarly, as in 4.4, this implies γ is $H(V)$-convex. But then γ as a subset of U is $H(U)$-convex. For if $x \in U - \gamma$ and $x \in V$, there is an $f \in H(V)$ such that $|f(x)| > \|f\|_\gamma$. But f is just the restriction to V of a function $g \in H(U)$, so $|g(x)| > \|g\|_\gamma$. If

$x \notin V$, then there is an $f \in H(U)$ such that $f \mid V = 0$, and $f(x) \neq 0$. Then $|f(x)| > \|f\|_\gamma$. Then, since γ is $H(U)$-convex, $H(U)$ is dense in $A(\gamma)$. But U is polynomial convex, implying that the polynomials are dense in $H(U)$ in the topology of uniform convergence on compact subsets of U, so the polynomials are dense in $A(\gamma)$. But by 4.5, $A(\gamma) = C(\gamma)$.

5. **The maximality theorem.** Before proceeding to the extension of Wermer's maximality theorem we prove a general Banach algebra result which appears in another form in [7].

DEFINITION. Let A be a function algebra on X. Let $Y \subset S(A)$. A is (Y, X)-*maximal* if and only if, for any algebra B, $A \subset B \subset C(X)$, $S(B) \supset Y$ implies $B = A$. We say A is a *maximal subalgebra of* $C(X)$ if A is (X, X)-maximal. This is the same as the definition which appears in [11].

5.1. THEOREM. *Let M be a maximal subalgebra of $C(X)$. Let $A \subset M$ be a closed subalgebra of M such that $A \supset I$, I an ideal of M. Let H be the hull of I in $S(M)$, $\mathcal{I} = \{f \text{ in } C(X) \text{ vanishing on } H \cap X\}$. Then if B is a closed subalgebra of $C(X)$, and $B \supset A$, then either $B \subset M$ or $B \supset \mathcal{I}$.*

Proof. Suppose there exists g in $B - M$. Let $\mathcal{P} = \{$polynomials in g, coefficients in $M\}$; then $\mathcal{P}$ is dense in $C(X)$. Let h be in $C(X)$, p_n in $\mathcal{P}$, such that p_n converges uniformly to h. Then for f in I, fp_n converges to fh. Since fp_n is in B, fh is in B. Thus B contains $\{fh; f \text{ in } I, h \text{ in } C(X)\}$. But then B contains the closed algebra generated by this set, which is a closed ideal of $C(X)$, obviously, $\mathcal{I}$, since it has the same hull as $\mathcal{I}$.

5.2. COROLLARY. *Let M be a maximal subalgebra of $C(X)$. Let $H_1, \cdots, H_n$ be pairwise disjoint hulls in $S(M)$. Let $A = \{f \in M; f \mid H_j \text{ is constant}, j = 1, \cdots, n\}$. Then A is a maximal subalgebra of $C(\Gamma(A))(\Gamma(A) \text{ is the Šilov boundary of } A)$, among all subalgebras having $S(A)$ as space of maximal ideals, i.e., A is $(S(A), \Gamma(A))$-maximal.*

Proof. A is a subalgebra of M, thus there exists a map $\pi: S(M) \to S(A)$. Obviously π is a homeomorphism on $S(M) - (\bigcup_{i=1}^n H_i)$ and $\pi(H_i) = y_i$, a single point, and $\pi(x) \supset \Gamma(A)$. Let $B \subset C(\Gamma(A))$, and $S(B) = S(A)$. B is a closed subalgebra of $C(X)$; thus either $B \subset M$, or $B \supset \{f \text{ in } C(X); f \mid_{(\bigcup H_i) \cap X} = 0\}$, since $\bigcup H_i$ is a hull, and we can apply the theorem. If $B \subset M$, since $S(B) = S(A)$, any f in B is such that $f \mid H_j = f(y_j) = $ constant. Thus $B \subset A$, so $B = A$. If not, then B, as a subalgebra of $C(\Gamma(A))$, contains $\{f \text{ in } C(\Gamma(A)); f(y_i) = 0 \text{ for } y_i \text{ in } \Gamma(A)\}$. But then, since B separates $\Gamma(A)$, obviously $B = C(\Gamma(A))$.

5.3. COROLLARY. *Let M be a maximal subalgebra of $C(X)$. Let $H_1, \cdots, H_n$ be disjoint hulls in $S(M)$, with $H_j \cap X \neq \varnothing$; $j = 1, \cdots, n$. Then $A = \{f \text{ in } M; f \mid_{H_j \cap X} \text{ is constant}, j = 1, \cdots, n\}$ is a maximal subalgebra of $C(\Gamma(A))$.*

Proof. We calculate the space of maximal ideals as above. Letting $H_j' = \text{hull (kernel } (H_j \cap X))$ we have $\pi: S(M) \to S(A)$ onto, $\pi: S(M) - \bigcup_{i=1}^n H_j'$

is a homeomorphism, $\pi(H_i) = y_i$, $\pi(X) = \Gamma(A)$. Let B be a subalgebra of $C(\Gamma(A))$, $B \supset A$. If $B \subset M$, since also $B \subset C(\Gamma(A))$, we must have for all f in B, $f|_{H_{j'}} = $ constant. Then, by definition of A, $B \subset A$, so $B = A$. If $B \not\subset M$, then as in 5.2, $B = C(\Gamma(A))$.

5.4. Corollary. *Let M be a maximal subalgebra of $C(X)$, A a subalgebra of M. Let $B \supset A$ be a subalgebra of $C(X)$. If A contains an ideal I of M such that* (hull $I) \cap X$ *is a finite point set, then $B \subset M$ or $B = C(S(B))$.*

Proof. Let (hull $I) \cap X = \{x_1, \cdots, x_n\}$. Then if B is not contained in M, by the theorem, $B \supset \{f \in C(X); f(x_i) = 0, i = 1, \cdots, n\}$. Since the Šilov boundary of B is an identification space of X, we can say that for points $a_1, \cdots, a_k$ in $S(B)$, $B \supset \{f \in C(S(B)); f(a_i) = 0\}$. Since $B|\{a_i\}$ is separating, it is $C(\{a_i\})$. Then if f is in B, there exists a g in B such that $g(a_i) = (f(a_i))^-$. Then $\bar{f} - g$ is in $C(S(B))$ and is 0 on $\{a_i\}$, so is in B. Then $\bar{f} = \bar{f} - g + g$ is in B. Thus with every f in B, its conjugate $\bar{f}$ also is in B; thus by the Stone-Weierstrass theorem, $B = C(X)$.

5.5. Theorem. *Let S be a connected one-dimensional analytic space, let K be compact, $K = \Omega \cup \gamma_1 \cup \cdots \cup \gamma_n$, Ω a domain, $\gamma_1, \cdots, \gamma_n$ piecewise analytic curves.*

(A) *$A(K)$ is a maximal algebra in $C(\partial K)$ if and only if* int $K = \Omega$ *is a complex manifold.*

(B) *In any case, $A(K)$ is contained in precisely one maximal algebra M, and if B is a subalgebra of $C(\partial K)$, $B \supset A(K)$, then B is dense in $C(\partial K)$ or $B \subset M$.*

Proof. Let R be the normal model for a neighborhood U of K, $\pi: R \to U$ the projection, so that $R = \bigcup_{i=1}^k U_i$, where U_i is a domain on a compact Riemann surface R_i. Let $\tilde{K}$ be as previously defined. Then $\tilde{K}$ is compact, and $\tilde{K} = \tilde{\Omega} \cup \tilde{\gamma}_1 \cup \cdots \cup \tilde{\gamma}_n$, where $\tilde{\Omega}$ is open and $\tilde{\gamma}_i$ are piecewise analytic curves, $\partial \tilde{K} = \tilde{\gamma}_1 \cup \cdots \cup \tilde{\gamma}_n$.

(1) We first prove the necessity of (A). Suppose $A(K)$ is a maximal subalgebra of $C(\partial K)$. Let $\{s_1, \cdots, s_n\} = K \cap S_{\text{sing}}$, and $\pi^{-1}(s_i) = \{x_{ij}; 1 \leq j \leq t_i\}$. Let $B = \{(f|\partial K) \circ \pi^{-1}; f$ in $A(\tilde{K}), f(x_{ij}) = f(x_{ik})$ for s_i in $\partial K\}$. Then B is a closed subalgebra of $C(\partial K)$, and since π is holomorphic, $B \supset A(K)$. Further, since every f in B has a holomorphic extension into $\tilde{\Omega}$, $B \neq C(\partial K)$. Thus, since by assumption $A(K)$ is maximal, $B = A(K)$. Now, for $x \neq y \in$ int $\tilde{K}$, there is a meromorphic function f on R, holomorphic on $\tilde{K}$, such that $f(x_{ij}) = 0$ for s_i in ∂K and $f(x) \neq f(y)$ (by 2.3). Thus, since $f \circ \pi^{-1}$ is in $B = A(K)$, there is a g in $A(K)$ such that $g \circ \pi = f$. Thus $g(\pi x) \neq g(\pi y)$, so $\pi x \neq \pi y$. Thus $\pi: \tilde{\Omega} \to \Omega$ is one-one, thus is a homeomorphism, so Ω has no identification points. Now let s be in int K, $\pi^{-1}(s) = x$ and g in $A(\tilde{K})$ such that $g(x_{ij}) = 0$ for s_i in ∂K, $g(x) = 0$ and $dg(x) \neq 0$. Then $g \circ \pi^{-1}$ is in $B = A(K)$; thus there is an f in $A(K)$ such that $f \circ \pi = g$. Now for some neighborhood $\tilde{U}$ of x, $g: \tilde{U} \leftrightarrow D$, a disc with center

the origin, so $f: \pi(\tilde{U}) = U \leftrightarrow D$ is a holomorphic homeomorphism. But f is biholomorphic in U. For let u be in Θ_t^s, t in U; then $\hat{\pi}(u)$ is in $\Theta_{\pi^{-1}(t)}^R$, and thus since g is biholomorphic, $\hat{g}^{-1}(\hat{\pi}(u))$ is in $\Theta_{g(\pi^{-1}(t))}^1$, i.e., $\hat{f}(u)$ is in $\Theta_{f(t)}^1$. Then (U, f) is a local parameter at s mapping U onto a manifold; this for all s in Ω, so Ω must be a manifold.

(2) We now prove the sufficiency of (A), i.e., we assume Ω is a manifold. Then $\Omega \cap S_{\text{sing}} = \varnothing$. Let

$$\{s_1, \cdots, s_n\} = K \cap S_{\text{sing}} = \partial K \cap S_{\text{sing}}, \text{ and } \pi^{-1}(s_i) = \{x_{ij}\}.$$

Since K is a Runge set, by 4.4, $A(K) = C(K) \cap H(\text{int } K)$, and also $\tilde{K}$ is a Runge set, so $A(\tilde{K}) = C(\tilde{K}) \cap H(\text{int } \tilde{K})$, which is (by Wermer's maximality theorem) a maximal subalgebra of $C(\partial \tilde{K})$. Then by 5.3, $M = \{f \text{ in } A(K);$ $f(x_{ij}) = f(x_{ik}), 1 \leq i \leq n\}$ is a maximal subalgebra of $C(Y)$, where $Y = \partial \tilde{K}$ with these points $\{x_{i1}, \cdots, x_{in_i}\}$ identified for $1 \leq i \leq n$. But then Y is homeomorphic to K via $\pi^{-1}: Y \leftrightarrow \partial K$, so $\{f \circ \pi^{-1}; f \text{ in } A(\tilde{K}), f(x_{ij}) = f(x_{ik}), 1 \leq i \leq n\}$ is a maximal subalgebra of $C(\partial K)$. But this is just $A(K)$. For surely it contains $A(K)$. Conversely, if f is in $A(\tilde{K})$, and $f(x_{ij}) = f(x_{ik})$, $1 \leq i \leq n$, then $f \circ \pi^{-1}$ is in $C(K)$. Further, $\pi: \tilde{\Omega} \leftrightarrow \Omega$ is biholomorphic (since $\Omega \subset S_{\text{reg}}$), so $f \circ \pi^{-1} | \Omega$ is in $H(\Omega)$. But $A(K) = C(K) \cap H(\Omega)$, then $f \circ \pi^{-1}$ is in $A(K)$. Thus $A(K)$ is a maximal subalgebra of $C(\partial K)$.

(3) We now prove (B). Let $\{s_1, \cdots, s_e\} \subset \partial K$, $\{s_{e+1}, \cdots, s_n\} \subset \text{int } K$. Let $A^1 = \{f \in A(\tilde{K}); f(x_{ij}) = f(x_{ik}), 1 \leq i \leq e\}$. Then by 5.3, A^1 is a maximal subalgebra of $C(Y)$ with $Y = \partial \tilde{K}$ with $\{x_{i1}, \cdots, x_{in_i}\}$ identified for $1 \leq i \leq e$. But then, as in the above Y is ∂K and $M = \{f \circ \pi^{-1}; f \in A^1\}$ is a maximal subalgebra of $C(\partial K)$. As we have observed, $\{f \circ \pi; f \in A(K)\}$ contains an ideal of $A(\tilde{K})$ whose hull is a finite point set. Thus $A(K)$ contains an ideal of M whose hull is a finite point set. Then by 5.4, if $B \subset C(\partial K)$ and $B \supset A(K)$, then either $B = C(S(B))$ or $B \subset M$. In the former case, since $A(K)$ separates points on ∂K, so does B, so $S(B) = \partial K$. Thus Theorem 5.5 is proven.

6. Extension of Wermer's theorem. In order to prove Theorem 6.7 it is necessary to appeal to several theorems of Oka.

6.1. THEOREM (ABBILDUNGSATZ). *Let S_1, S_2 be analytic spaces and g a proper holomorphic map of S_1 into S_2. Suppose also that $g^{-1}(x)$ is finite for all $x \in X_2$. Then $g(S_1)$ is a variety in S_2* [8].

6.2. THEOREM. *Let K be a polynomial-convex subset of C^n. Then the polynomials are dense in $A(K)$* [5].

6.3. THEOREM. *Let V be a variety in the $H(U)$-convex domain U in C^n. Let $f \in H(V, \Theta^v)$. Then there is a $g \in H(U, \Theta^n)$ such that $g | V = f$* [5].

We shall also use the following lemmas concerning various types of convexities.

6.4. **Lemma.** *Let V be a variety in a polynomial convex domain P in C^n, $K \subset V$ a compact $H(V, \Theta^V)$-convex set. Then K is polynomial convex.*

Proof. Let $x \notin K$. If x is not in V, there is an $fH(P, \Theta^n)$ such that $f(x) = 1$, $f|V = 0$. If x is in V, there is an f in $H(V, \Theta^V)$ such that $f(x) = 1$ and $\|f|K\| < \epsilon$. But by 6.3 $f = g|V$; $gH(P, \Theta^n)$, so $g(x) = 1$ and $\|g|K\| < \epsilon$. Thus K is $H(P, \Theta^n)$-convex, but since P is polynomial convex, K is also polynomial convex.

6.5. **Lemma.** *Let S be a connected set in C^n satisfying: for all $x \in S$ there is a neighborhood U_x of x such that $S \cap U_x$ is a variety in U_x (i.e., S is locally a variety). Suppose $K \subset S$, as a subset of C^n is compact and polynomial convex. Then for any $f \in H(U \cap S, \Theta^S)$, U a neighborhood of K, $f|K$ is uniformly approximable by polynomials.*

Proof. For x in K, there is a U_x such that $S \cap U_x$ is a variety in U_x. Cover K by finitely many $U_1, \cdots, U_n$. Let V_j open, $\overline{V}_j$ compact, contained in U_j and $V_1, \cdots, V_n$ cover K. Then $S \cap \overline{V}_j$ is compact. Let $K' = \bigcup_{j=1}^n S \cap \overline{V}_j$. Then $K' \subset S$, K' is compact and int $K' \supset K$. By the polynomial convexity, we can find an analytic polyhedron $P = \{|p_j| < 1; j = 1, \cdots, k, p_j$ polynomials$\}$ such that $K \subset P \cap S \subset$ int K'. Then $P \cap S$ is closed in P (for K' is closed, and $K' \cap P = S \cap P$). Thus since $P \cap S$ is locally a variety, $P \cap S$ is a variety in P. Every f in $H(P \cap S, \Theta^S)$ then is the restriction of g in $H(P, \Theta^n)$ to $P \cap S$ by 6.3. Since P is polynomial convex, $g|_K = f|_K$ is approximable by polynomials, by 6.2.

6.6. **Lemma.** *K is a compact subset of a Riemann surface. $f_1, \cdots, f_n$ are holomorphic in a neighborhood U of K, and $\omega: U \to S \subset C^n$, $\omega(r) = (f_1(r), \cdots, f_n(r))$, where S is locally a variety. Suppose K is convex with respect to polynomials in $f_1, \cdots, f_n$. Then $\omega(K)$ is polynomial convex.*

Proof. Let $U \supset \overline{V} \supset V \supset K$, V a domain, $\overline{V}$ compact. Then $\omega(\overline{V})$ is compact. For x in $\partial \overline{V}$, there is a $p_x(f_1, \cdots, f_n)$ such that $|p_x(x)| > \|p_x\|_K$, p_x a polynomial. Then $|p_x(\omega(x))| > \|p_x\|_{\omega(K)}$. Thus for every z in $\omega(\overline{V})$, there is a polynomial p_z such that $|p_z(z)| > \|p_z\|_{\omega(K)}$. By compactness, there are polynomials $p_1, \cdots, p_k$ such that $\omega(\overline{V}) \supset \{|p_j| < 1, j = 1, \cdots, k\} \cap S \supset \omega(K)$. If we add proper multiples of the coordinate functions, we can say that there is an analytic polyhedron $P = \{|p_j| < 1; p_j$ polynomials$\}$ such that $K \subset P$ and $P \cap S$ is a variety in P. $P \cap S$ is closed in P, for it is the intersection of a closed set, $\omega(\overline{V})$, with P. Now $\mathfrak{a}(H(S \cap P, \Theta^S))$ contains the polynomials in $f_1, \cdots, f_n$, so by assumption, $\omega(K)$ is $H(S \cap P, \Theta^S)$-convex. Now Lemma 6.4 applies, proving $\omega(K)$ is polynomial convex.

6.7. **Theorem.** *Let K be a compact subset of a Riemann surface R, and let U be a neighborhood of K. Suppose A^0 is a subalgebra of $H(U)$ such that: (i) A^0 is generated by $f_1, \cdots, f_n \in H(U)$; (ii) K is A^0-convex; (iii) $\{p \in U;$ there is a $q \in U$ such that $p \neq q$ but $f(p) = f(q)$, for all $f \in A^0\}$ is finite.*

Let A be the uniform closure of A^0 on K. Then there is a variety V in the unit polycylinder of some C^n and a map ϕ of a neighborhood of K onto V such that

$$A = \{f \circ \phi; f \text{ in the closure of the polynomials on } \phi(K)\}.$$

Proof. We note first that if K is a domain bounded by an analytic arc γ and $df_1 \neq 0$ on γ, that (ii) follows from conditions (i) and (iii) by Wermer's Theorem (1.1). On the other hand, the methods of the proof are general for any analytic space, so the theorem still holds if we replace R by any analytic space (of any dimension).

Because of (ii) we can find $p_1, \cdots, p_t \in A^0$ such that $\{r \in R; |p_i(r)| \leq 1, 1 \leq i \leq t\} = \overline{W}$ is compact on R. By multiplication of the f_i by a constant we may assume $\|f_i\|_{\overline{W}} < 1$, $1 \leq i \leq n$. Then, easily, the map ϕ on $W = \{r \in R; |p_i(r)| < 1, 1 \leq i \leq t\}$,

$$\phi(r) = (f_1(r), \cdots, f_n(r), p_1(r), \cdots, p_t(r)),$$

is a proper holomorphic map of W into $P^{n+t} = \{w; |w_j| < 1\}$, and $\phi^{-1}(w)$ is finite for all w. Thus, by 6.1, $\phi(W) = V$ is a subvariety of P^{n+t}, and $\phi(K)$ is a compact subset of $\phi(W)$. Now since K is A^0-convex, it is $H(W)$-convex, and thus $\phi(K)$ is $H(V)$-convex. Thus by 6.4 $\phi(K)$ is polynomial convex. It follows that the polynomials are dense in $A(\phi(K))$.

Thus if $f \in A(\phi(K))$, f is approximable on $\phi(K)$ by polynomials p_n. Then $p_n \circ \phi | K$ converges to $f \circ \phi$. But $p_n \circ \phi = p_n(f_1, \cdots, f_n, p_1, \cdots, p_t)$ is a polynomial in $f_1, \cdots, f_n, p_1, \cdots, p_t$, and thus is in A^0 since $f_1, \cdots, f_n, p_1, \cdots, p_t \in A^0$. Thus $f \circ \phi \in A$. Conversely, if $f \in A$, we have $g_n | K$ converging to f with $f_n \in A^0$, in particular $g_n = p_n(f_1, \cdots, f_n)$ where p_n is a polynomial. Thus $g_n = p_n \circ \phi$, so that $f_n \in \{f \circ \phi; f \text{ a polynomial on } \phi(K)\}$, so f is in the closure of this set, which is $\{f \circ \phi; f \text{ in the closure of the polynomials on } \phi(K)\}$. Thus 6.7 is proven.

If K is, in particular, a domain with finitely many piecewise analytic boundary curves, then for any algebra A^0 satisfying only (ii) and (iii) we can obtain the desired conclusion: that its closure on K can be represented as $A(K^0)$ for K^0 lying on some analytic space. In this case we can find finitely many functions $f_1, \cdots, f_n$ satisfying the hypotheses of 6.7. Thus the algebra A^1 generated by $f_1, \cdots, f_n$ on K can be represented as $A(K^1)$, where K^1 is a compact set on an analytic space V. Because only finitely many points can be identified, K^1 is also a domain bounded by piecewise analytic curves. It is clear that R is the normalization for V, and A, the closure of A^0 on R is a subalgebra of $A(K)$ containing $A(K^1)$. Thus A is of finite codimension in $A(K)$, which will guarantee that A is an $A(K^0)$ on some analytic space.

6.8. COROLLARY. *Let $f_1, \cdots, f_n$ be functions holomorphic in an annulus $\mathfrak{a}$ about $\{|z| = 1\} = \Gamma$. Suppose $f_1, \cdots, f_n$ separate the points of Γ, df_1 is never zero on Γ, and the algebra of $A^0 = $ uniform closure on Γ of polynomials in*

$f_1, \cdots, f_n$ is not $C(\Gamma)$. *Then there is a connected set $S \subset C^n$, a closed analytic curve γ, bounding an open connected subset Ω of S, and a homeomorphism $\phi: \Gamma \leftrightarrow \gamma$ such that*: (i) S *is locally a variety*; (ii) $A^0 = [f \circ \phi; f$ *in the closure of polynomials on $\Omega \cup \gamma]$.*

Proof. Under the assumption Wermer has proven: there exists a Riemann surface R, a biholomorphism $\phi': \alpha \leftrightarrow \alpha' \subset R$, $\phi'(\Gamma) = \gamma'$, an analytic curve bounding a domain Ω' such that for $A_1^0 = [f \circ \phi^{-1}; f$ is in $A^0]$, $A_1^0 \subset A(\Omega' \cup \gamma')$ and $A_1^0 \supset \mathscr{I}$, $\mathscr{I}$ an ideal of $A(\Omega' \cup \gamma')$, with finite hull in Ω'.

Let F_i be the function in $A(\Omega' \cup \gamma')$ which extends $f_i \circ \phi^{-1}$ to all of $\Omega' \cup \gamma'$. Then $F_1, \cdots, F_n$ are holomorphic in a neighborhood U of $\Omega' \cup \gamma'$. Let $P = \{$polynomials in $f_1, \cdots, f_n\}$. Then P is a subalgebra of $H(U)$, and is dense in A_1^0. Then, since A_1^0 contains an ideal with finite hull contained in Ω', P satisfies (ii) and (iii) of 6.5 and thus we can apply Theorem 6.7 to A^0.

Appendix. We shall prove a special case of Wermer's Theorem, 1.1. Although this special case in a sense avoids the greatest difficulty in the theorem, that of proving convexity, still it is fairly general and indicates how a proof using the theory of several complex variables might go.

THEOREM. *Let M be a compact Riemann surface, $K = \Omega \cup \gamma$, Ω a domain and γ an analytic curve. Let $f_1, \cdots, f_n$ be meromorphic in M and holomorphic on K such that*

$$N = \{p \text{ in } M; \text{ there is } q \text{ in } M \text{ such that } f_j(p) = f_j(q) \text{ for all } j = 1, \cdots, n\}$$

is finite. Let $A^0 = $ uniform closure on K of the algebra of polynomials in $f_1, \cdots, f_n$. Then A^0 contains an ideal of $A(K)$ of finite codimension.

Proof. We may assume $\|f_j\|_K < 1$ for $j = 1, \cdots, n$. Let $R = \{m$ in M; $|f_j(m)| < 1\}$, $\overline{R}$ is compact since the f_j are meromorphic on M. Define $F: R \to P^n = \{z; |z_j| < 1, 1 \leq j \leq n\}$,

$$F(m) = (f_1(m), \cdots, f_n(m)).$$

(1) F is holomorphic and it is proper. For let m_n be in $F^{-1}(A)$, A compact in P^n. $\overline{R}$ is compact, so there exists an m in $\overline{R}$ such that $m_{n_k} \to m$. Then $f_i(m)$ is a limit point of $\{f_i(m_n)\}$ and since $\{f_i(m_n)\} = \{\pi_i F(m_n)\}$ is contained in a compact subset of the unit disc, we have $|f_i(m)| < 1$. Therefore, $F(m) = (f_1(m), \cdots, f_n(m))$ is defined and is in P^n; since A is compact, $F(m)$ is in A. Therefore m in $F^{-1}(A)$; and $F(A)$ is compact.

Now because N is finite, we have that $F^{-1}(x)$ is always finite for $x \in P^n$. Thus, by the Abbildungsatz (6.1), $F(R) = S$ is a variety in P^n.

(2) (R, F) is the normalization for S.

Let s be in S. Let $F^{-1}(s) = \{x_1, \cdots, x_k\}$. Each x_i has a neighborhood U_i in S such that $(U_i)^- \cap N$ is empty, thus F is one-one on $(U_i)^-$, and $(U_i)^-$ is compact. Thus $F: U_1 \leftrightarrow F(U_i)$ is a homeomorphism. Since $\bigcup_{i=1}^n \{df_i = 0\}$ is dis-

crete, we may also assume that except possibly for x_i, some $df_i \neq 0$ at every point of U_i. Thus $F\big|_{(U_i - x_i)}$ is a holomorphic map of a connected set onto $F(U_i) - s$, so $F(U_i) - s$ is a connected submanifold of S_{reg} of dimension one, and thus must be an open set contained in one irreducible branch S_i of S at s. Thus $F(U_i)$ is a neighborhood of s on S_i. Then $\bigcup_{i=1}^{n} F(U_i)$ is a neighborhood of s. For suppose $s_n \to s$, s_n is S. Let x_n be in R such that $F(x_n) = s_n$; then by the properness of F there exists an x in R such that $x_n \to x$, so x must be some x_i, since $F(x) = s$. But then the x_n from some n^0 on must be in U_i, so s_n in $F(U_i)$. Thus $\bigcup_{i=1}^{n} F(U_i)$ is a neighborhood of s on S.

Let s be in S_{sing}. The branches of S at s are just the $F(U_i)$ described above, then s is an identification point if and only if $F^{-1}(s)$ is more than one point. If $F^{-1}(s) = x$, then we must have here $df_j(x) = 0$, $1 \leq j \leq n$, for if not then F is biholomorphic in a neighborhood of x, which would make S a manifold at s. Conversely, if s is in S_{reg}, then $dz_j(s) \neq 0$ for some j, i.e., z_j is one-one in a neighborhood of S, implying $df_j(x) \neq 0$. Thus $F^{-1}(S_{\text{sing}}) = N \cup \{df_j = 0, 1 \leq j \leq n\}$, which is a finite set (since $\overline{R}$ is compact). Now, for x not in $F^{-1}(S_{\text{sing}})$ we have that some $df_j \neq 0$, and F is a homeomorphism in a neighborhood of x onto a neighborhood of $F(x)$; it follows that F is biholomorphic in a neighborhood of x. Thus $F\big|_{R - F^{-1}(S_{\text{sing}})}$ is biholomorphic. Thus (R, F) is the normal model for S.

It is easy to see that K, since it consists of a domain bounded by one analytic curve, can be written as the decreasing intersection of domains simply connected with respect to R. Then, by 4.1, K is $H(R)$-convex. Then $K^0 = F(K)$ is also $H(S)$-convex, as in Theorem 4.4, for in this case $K = \tilde{K}^0$. Then by 6.4, K^0 is polynomial convex.

Now, as in §5, $\{f \circ F; f \text{ in } A(K^0)\}$ contains an ideal I of $A(K)$ of finite codimension. If f is in $H(U, \Theta^S)$, U a neighborhood on S of K, then by 6.3, there are polynomials p_n such that $p_n\big|_{K^0} \to f\big|_{K^0}$. Thus $p_n \circ F \to f \circ F$ on K, but $p_n \circ F = p_n(f_1, \cdots, f_n)$ is in A^0. Thus $f \circ F$ is in A^0, and finally $I \subset A^0$ (in fact, by 6.7, $\{f \circ F; f \text{ in } A(K)^0)\} = A^0$).

COROLLARY. *If in addition we assume that $f_1, \cdots, f_n$ separate the points of K and for all m in K there is an f_j such that $df_j(m) \neq 0$, then $A^0 = A(K)$.*

Proof. In this case there is a neighborhood U of K such that $F\big|_U$ is biholomorphic. Then $\hat{F}(H(F(U), \Theta^S)) = H(U, \Theta^R)$. Since f in the former is approximable on K by polynomials in $f_1, \cdots, f_n$, then A^0 is dense in $H(U, \Theta^R)$. The latter is dense in $A(K)$; thus $A^0 = A(K)$.

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